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THIRD HOMOLOGY OF SL(2, $\mathbb{R}$) MADE DISCRETE*

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Let G^δ denote the discrete group associated to a Lie group G. Let B denote the classifying space functor associated to topological groups. The Eilenberg-MacLane homology $H_*(G, \mathbb{Z})$ can be identified with $H_*(BG^\delta, \mathbb{Z})$. Based on suggestions made by Friedlander concerning the Lichtenbaum-Quillen conjecture on the cohomology of $\mathrm{GL}(\infty, F)$ for an algebraically closed field F, Milnor [11] posed the following conjecture:

The natural map $H_*(BG^\delta, \mathbb{F}_p) \to H_*(BG, \mathbb{F}_p)$ is an isomorphism for all primes p.

As evidence, Milnor showed that the conjecture depends only on the Lie algebra of G and verified it for solvable groups. The general case is then reduced to the case where G is connected, simple and nonabelian (and simply connected whenever convient). Under these added hypotheses, Milnor showed that the natural map is always surjective. Most of the additional evidence supporting the conjecture occurs on the level of H_2 and is based on 'K_2-calculations' or Schur multiplier calculations due to Steinberg, Matsumoto-Moore, see [10, 17], and Deodhar [3]. In a separate work [14], a beautiful theorem of Mather, see [8] or [1], has been extended so that the above conjecture is correct for $H_2(G, \mathbb{F}_p)$ where G is any compact Lie group of classical type. Beyond the level of H_2, precise evidence becomes very sparse. In fact, the only known case is that of $H_3(\mathrm{SL}(2, \mathbb{C}), \mathbb{F}_p)$ found in Dupont-Sah [4].

The present work is a variation of Dupont-Sah [4]. The fact that $\mathbb{R}$ is not algebraically closed and the fact that $\mathrm{SL}(2, \mathbb{R})$ is not simply connected both cause some difficulties. The result we obtain is not as strong as that in Dupont-Sah. Namely, we are only able to show that $H_3(\mathrm{SL}(2, \mathbb{R}), \mathbb{Z})$ is 2-divisible, or $H_3(\mathrm{SL}(2, \mathbb{R}), \mathbb{F}_2) = 0$, and thereby confirming the conjecture of Milnor in this case.

To be more precise, we take advantage of the ordering of $\mathbb{R}$ to define a simplicial

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complex W with $\mathrm{SL}(2,\mathbb{R})$ acting simplicially. This is a variation of the idea of Bloch–Wigner that was worked out in Dupont–Sah [4]. $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$ is then shown to be an abelian group generated by 'cross-ratio' symbols: $\{r\}$, $r>1$, with defining relations:

$$\{r_1\}-\{r_2\}+\{r_2/r_1\}-\{(r_2-1)/(r_1-1)\}+\{(1-r_2^{-1})/(1-r_1^{-1})\}=0,$$
$$1<r_1<r_2 \quad \text{in } \mathbb{R}.$$

Using Rogers' L-function [12], $\{2\}$ is shown to generate an infinite cyclic subgroup of $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$ and $48\cdot\{2\}$ then generates the subgroup corresponding to the infinite direct summand of $H_2(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ with generator $c(-1,-1)$. This latter arises from the fundamental group of $\mathrm{SL}(2,\mathbb{R})$. The number 48 appears suspiciously similar to the order of the cyclic group $K_3(\mathbb{Z})$. This use of Rogers' L-function is similar to the use of the dilogarithm by Bloch–Wigner [2], Dupont–Sah [4], Gelfand–MacPherson [5], and Wigner [18]. It should be emphasized that the defining relations are the basic objects of our concern. This is the same philosophy as that of the various references just mentioned. However, our approach is more primitive. Specifically, we have not touched the geometry of the space obtained from W through a 'filling-in' process, compare Gelfand–MacPherson [5]. The group $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ appears as a subgroup of $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)/\mathbb{Z}\cdot 48\{2\}$ in the form of the kernel of a 'Dehn-invariant' homomorphism and the quotient is a suitable $\mathbb{Q}$-subspace of the $\mathbb{Q}$-vector space $\Lambda^2_{\mathbb{Z}}(\mathbb{R}^+)$. This fits with the description in Dupont–Sah [4], as well as Sah–Wagoner [15]. The 2-divisibility of $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ then follows from the 2-divisibility of $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$. According to Milnor's conjecture, $H_i(W/\mathrm{SL}(2,\mathbb{R})^\delta)$ should be a $\mathbb{Q}$-vector space for $i\geq 3$ An equivalent formulation was suggested by Wu-Chung Hsiang for a complex of the same (weak) homotopy type as $W/\mathrm{SL}(2,\mathbb{R})^\delta$. We show that the known torsion in $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ leads to a direct summand isomorphic to $\mathbb{Q}$ in $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$ with $\{2\}$ as a generator. It should be noted that the usual number theoretic construction of discrete cofinite volume subgroup of $\mathrm{SL}(2,\mathbb{C})$ is not available to generate elements of $H_3(\mathrm{SL}(2,\mathbb{R}))$ so that nontorsion elements of $H_3(\mathrm{SL}(2,\mathbb{R}))$ would have to be constructed by other means (see Appendix B). A tempting conjecture on the structure of $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$ would be that the Dehn-invariant and the homomorphism based on Rogers' L-function detect $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$. Testing of this conjecture in terms of the presentation of $H_3(W/\mathrm{SL}(2,\mathbb{R})^\delta)$ leads to many curious questions that we hope to consider elsewhere.

It should be apparent that the present work owes much to J.L. Dupont, as well as S. Bloch and D. Wigner. In addition, conversations with Wu-Chung Hsiang convinced us that a detailed study of $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ should be carried out in order to reveal some of the difficulties that have to be resolved in tackling the conjecture of Milnor (at least for $\mathrm{SL}(n,\mathbb{R})$). Finally, one of the arguments used was worked out during a conversation with A.T. Vasquez.We wish to thank each of them for the inspirations they provided us.

1. A simplicial complex for SL(2, $\mathbb{R}$)

Let $G = \mathrm{SL}(2, \mathbb{R})$. Let $S(\mathbb{R}^2)$ denote the space of all rays in $\mathbb{R}^2$ emanating from the origin. $S(\mathbb{R}^2)$ therefore doubly covers the projective line $\mathbb{P}^1(\mathbb{R})$. We will parametrize a precise fundamental domain for this double covering by the set $\mathbb{R} \cup \{-\infty\}$. Namely, the ray $\mathbb{R}^+ \cdot \binom{-1}{0}$ will be labelled by $-\infty$ and the ray $\mathbb{R}^+ \cdot \binom{r}{1}$ will be labelled by $r \in \mathbb{R}$. This parametrization of $\mathbb{P}^1(\mathbb{R})$ or more precisely of a semicirle will be called the slope parametrization. It can also be parametrized by $\theta = \arctan r \in [-\pi/2, \pi/2)$. This latter will be called the radian parametrization. We note that the universal covering space of $\mathbb{P}^1(\mathbb{R})$ or of $S(\mathbb{R}^2)$ is simply $\mathbb{R}$. We can identify these spaces through the Iwasawa decomposition $G = KAN$ with $K = \mathrm{SO}(2, \mathbb{R})$, $A \cong \mathbb{R}^+$ is the group of diagonal matrices with positive diagonal entries, and N is the group of all upper unipotent matrices so that $N \cong \mathbb{R}$. We have the following central extension for the universal covering group $\tilde{G}$ of G:

$$(1.1) \qquad 0 \to \pi_1(G) \to \tilde{G} \to G \to 1.$$

The Iwasawa decomposition of G can be lifted to the following decomposition of $\tilde{G}$:

$$(1.2) \qquad \tilde{G} = \mathbb{R}AN.$$

The extension (1.1) splits on the subgroup AN of G. The connectivity of AN shows that the splitting is unique. As a result, we can identify AN in a unique manner with a subgroup of $\tilde{G}$. $S(\mathbb{R}^2)$ can be identified with K or with the coset space G/AN and its universal covering space can be identified with $\tilde{G}/AN$. We note that the center of $\tilde{G}$ is an infinite cyclic group containing $\pi_1(G)$ as a subgroup of index 2. The generator of $\pi_1(G)$ is usually denoted by the symbol $c(-1, -1)$, see Sah-Wagoner [15].

$$(1.3) \qquad \pi_1(G) \cong \mathbb{Z} \cdot c(-1, -1).$$

The symbol $c(-1, -1)$ is mapped onto the symbol $\{-1, -1\}$ in $K_2(\mathbb{R})$. The latter is the unique element of order 2 in $K_2(\mathbb{R})$.

Let W be the simplicial complex formed by all nonempty finite subsets $\{v_0, \ldots, v_i\} \subset S(\mathbb{R}^2)$, satisfying the following convexity condition:

(1.4) $\quad v_0, \ldots, v_i$ are contained in some (varying) open half plane.

Let $C_*(W)$ denote the complex of *alternating* simplicial chains on W. Using (1.4), we see that:

(1.5) $\quad C_i$ is a free abelian group based on $(v_0, \ldots, v_i)$ such that v_s precedes v_t for $s < t$ in the counterclockwise orientation of $S(\mathbb{R}^2)$ and $0 < \arg v_i - \arg v_0 < \pi$ when $i > 0$.

Evidently, $G = \mathrm{SL}(2, \mathbb{R})$ acts simplicially on W through its action on $S(\mathbb{R}^2)$. It is easy to see that:

(1.6) C_i is $\mathbb{Z}G$-free for $i \geq 2$. C_2 is isomorphic to the group ring of G using the generator $(-\infty, 0, 1)$ in the slope parametrization. C_i is $\mathbb{Z}K$-free for $i \geq 0$ with C_0 isomorphic to the group ring of K using the generator $(-\infty)$.
$C_1 \cong \operatorname{ind}_A^G \mathbb{Z} \cdot (-\infty, 0), \qquad C_0 \cong \operatorname{ind}_{AN}^G \mathbb{Z} \cdot (-\infty)$.

There is no problem constructing a simplicial complex $\tilde{W}$ covering W with $\pi_1(G)$ as the group of covering transformations. Namely, $\tilde{W}$ is just the simplicial complex of all nonempty finite subsets $(\theta_0, \ldots, \theta_i)$ of real numbers satisfying the condition:

$$\theta_0 < \cdots < \theta_i, \qquad 0 < \theta_i - \theta_0 < \pi \quad \text{when } i > 0. \tag{1.7}$$

The projection map is then induced from the covering map: $\mathbb{R} \to S(\mathbb{R}^2)$. Let $C_*(\tilde{W})$ denote the *alternating* simplicial chain complex associated to $\tilde{W}$. It is immediate that:

$$C_*(W) \cong C_*(\tilde{W}) \otimes_{\mathbb{Z}\pi_1(G)} \mathbb{Z}, \qquad \pi_1(G) \text{ acts freely on } \tilde{W}. \tag{1.8}$$

1.9. Proposition. *The chain complex $C_*(\tilde{W})$ is acyclic with augmentation $\mathbb{Z}$. $C_*(W)$ has the integral homology of the circle $S(\mathbb{R}^2)$. $C_*(W) \otimes_{\mathbb{Z}G} \mathbb{Z}$ is 2-acyclic with augmentation $\mathbb{Z}$ and can be identified with the complex of cellular chains on a cell complex $W/G^\delta \cong \tilde{W}/\tilde{G}^\delta$.*

Proof. Using the linear structure of $\mathbb{R}$, each simplex of $\tilde{W}$ can be identified with a unique singular simplex of $\mathbb{R}$. This identifies $\tilde{W}$ with a subcomplex of the total singular complex of $\mathbb{R}$. Using barycentric subdivision and simplicial approximation, it is easy to see that $\tilde{W}$ is chain equivalent to the total singular complex of $\mathbb{R}$. This yields the acyclicity of $C_*(\tilde{W})$. We note that the identification described is only compatible with the subgroup $\mathbb{R}$ of $\tilde{G}$.

Since $C_*(\tilde{W})$ is acyclic and is $\mathbb{Z}\pi_1(G)$-free, it follows that the homology of $C_*(\tilde{W}) \otimes_{\mathbb{Z}\pi_1(G)} \mathbb{Z}$ is just the Eilenberg-MacLane homology of $\pi_1(G) \cong \mathbb{Z}$. Since the circle $S(\mathbb{R}^2)$ is a $K(\mathbb{Z}, 1)$-space, we have the second assertion.

Since G^δ acts simplicially on W, there is no problem constructing a cell complex structure on W/G^δ compatible with the projection map. Same statement holds for $\tilde{G}^\delta$, $\tilde{W}$ as well as $\tilde{W}/\tilde{G}^\delta$. The homeomorphism between W/G^δ and $\tilde{W}/\tilde{G}^\delta$ follows from (1.1) and the fact that $W = \tilde{W}/\pi_1(G)$. The 2-acyclicity follows from the fact that W/G^δ has only one cell in dimensions 0, 1 and 2 as indicated in (1.6) and the boundary of the 2-cell is just the 1-cell. The identification of $C_*(W) \otimes_{\mathbb{Z}G} \mathbb{Z}$ as the complex of cellular chains is clear. □

At this point, the complex $C_*(\tilde{W})$ can be used to compute the integral homology of $\tilde{G}$ along well-known lines. Namely, the second index filtration of the double complex $C_*(\tilde{G}) \otimes_{\mathbb{Z}\tilde{G}} C_*(\tilde{W})$ leads to a spectral sequence $''E^1_{i,j}$ converging to $H_*(\tilde{G}, \mathbb{Z})$, see Serre [16, p. 95].

2. Homology of SL(2, ℝ) and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

We begin the study of $H_*(\widetilde{\mathrm{SL}}(2,\mathbb{R}),\mathbb{Z})$ through the spectral sequence mentioned at the end of Section 1. We set $G=\mathrm{SL}(2,\mathbb{R})$ and $\tilde G=\widetilde{\mathrm{SL}}(2,\mathbb{R})$. $''E^1_{i,j}$ is then $H_i(\tilde G, C_j(\tilde W))$ and can be displayed below:

$$
(2.1)\qquad
\begin{array}{llll}
\cdots & & & \\
C_3(\tilde W/\tilde G^\delta) & 0 & \cdots & \big\downarrow d^1=\varepsilon\partial_{\tilde W} \\
\mathbb{Z} & 0 & \cdots & \\
\mathbb{Z} & A & H_2(A,\mathbb{Z})\cdot(-\infty,0),\ldots,H_i(A,\mathbb{Z})\cdot(-\infty,0) & \\
\mathbb{Z} & A & H_2(A,\mathbb{Z})\cdot(-\infty),\ \ \ldots,H_i(A,\mathbb{Z})\cdot(-\infty) &
\end{array}
$$

Since $A\cong\mathbb{R}$ through the logarithm map, $H_i(A,\mathbb{Z})\cong\Lambda^i_{\mathbb{Z}}(\mathbb{R})$ is a $\mathbb{Q}$-vector space for $j>0$. The vanishing of $''E^1_{i,j}$ for $i>0$ and $j\geq 2$ follows from the fact that $C_j(\tilde W)$ is $\mathbb{Z}\tilde G$-free for $j\geq 2$. Compare (1.6) for the analogous statements for W and G. The description of $''E^1_{i,j}$ for $j=0$ and 1 follows from Shapiro's Lemma together with the analogous induced module structure described in (1.6). For $j=0$, we used center kills lemma together with the fact that AN is the semidirect product of $\mathbb{R}$ by $\mathbb{R}^+$. Note that $r\in\mathbb{R}^+\cong A$ acts on $N\cong\mathbb{R}$ through multiplication by r^2. Since the normalizer of A in $\tilde G$ induces inversion on A and covers a rotation of $\pi/2$ radians, the map d^1 from the row indexed by $j=1$ is multiplication by $(-1)^i-1$ on $\Lambda^i_{\mathbb{Z}}(\mathbb{R})$. We display the $''E^2_{i,j}$ terms:

$$
(2.2)\qquad
\begin{array}{llllll}
\cdots & & & & & \\
H_3(\tilde W/\tilde G^\delta) & 0 & \cdots & & d^2=0 & \\
0 & 0 & \cdots & & & \\
0 & 0 & \Lambda^2_{\mathbb{Z}}(\mathbb{R})\cdot(-\infty,0) & 0 & \Lambda^4_{\mathbb{Z}}(\mathbb{R})\cdot(-\infty,0),\ldots & \\
\mathbb{Z} & 0 & \Lambda^2_{\mathbb{Z}}(\mathbb{R})\cdot(-\infty) & 0 & \Lambda^4_{\mathbb{Z}}(\mathbb{R})\cdot(-\infty),\ldots &
\end{array}
$$

According to the conjecture in Milnor [11], the contractible Lie group $\tilde G$ should be finitely acyclic, i.e., it has the $\mathbb{F}_p$-homology of a point. It is evident from (2.2) that:

2.3. Proposition. *Equivalent statements are:*
(a) $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *is finitely acyclic.*
(b) *The cell complex* $\tilde W/\tilde G^\delta$ *or* W/G^δ *is finitely acyclic.*

In an oral communication, Wu-Chung Hsiang had conjectured the validity of (b) in Proposition 2.3 for a slightly different cell complex. It is not too difficult to show that his cell complex is (weakly) homotopically equivalent to ours. We will now consider the structure of $H_3(\tilde W/\tilde G^\delta)$. Conjecturally, $H_j(\tilde W/\tilde G^\delta)$ is a $\mathbb{Q}$-vector space for $j\geq 3$. We may work with either W/G^δ or $\tilde W/\tilde G^\delta$.

Using (1.6), it is immediately clear that $H_3(W/G^\delta)$ is generated by the 3-cells $(-\infty, 0, 1, r)$, $r>1$, in the slope parametrization and satisfying the defining relations:

$$\begin{aligned}(2.4)\qquad &\{r_1\}-\{r_2\}+\{r_2/r_1\}-\{(r_2-1)/(r_1-1)\}+\{(1-r_2^{-1})/(1-r_1^{-1})\}=0\\ &\text{where}\quad 1<r_1<r_2 \quad\text{and}\quad \{r\}=(-\infty,0,1,r),\quad r>1.\end{aligned}$$

We call $\{r\}$ the cross-ratio symbol of $(-\infty,0,1,r)$. The relation (2.4) arises from taking the boundary of $(-\infty,0,1,r_1,r_2)$. We remind ourselves about the rules on the action of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ on $C_3(W)$. The element

$$\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\ \text{in } A,\qquad ab=1$$

and a in $\mathbb{R}^+$ multiplies the slope of a ray by a^2. The element

$$\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\ \text{in } N$$

adds t to the slope. The usual rules governing $-\infty$ are valid. We only apply these rules to rays in the parametrized semicircle. The same actually applies to the half circle rotated by π radians through the application of $-I_2$. The validity of (2.4) is now straightforward.

In order to keep track of some of our calculations, we extend the symbol $\{r\}$ for r in $\mathbb{R}\cup\{-\infty\}$ according to the following rules:

$$(2.5)\qquad \{-\infty\}=\{0\}=\{1\}=0,$$

$$(2.6)\qquad \{r\}+\{r^{-1}\}=0,\quad r>0,$$

$$(2.7)\qquad \{-r\}=\{1+r^{-1}\},\quad r>0.$$

As can be seen, these are really nothing more than the fact that $C_*(W)$ is the alternating chain complex so that $(-\infty,r_1,r_2,r_3)$ for r_i in $\mathbb{R}$ should alternate with respect to permutations of r_1,r_2,r_3 and be 0 when duplication occurs.

With (2.5) through (2.7) in effect, it is easy to show that:

(2.8) (2.4) remains valid provided that $(r_1-1)(r_2-1)>0$, $r_1,r_2>0$.

We now consider $0<r_i<1$ and set $s_i=1-r_i$. Adding the relations (2.4) and using (2.6), (2.8), we obtain

$$(2.9)\qquad \{r_1\}+\{1-r_1\}=\{r_2\}+\{1-r_2\},\quad 0<r_i<1.$$

Combining (2.6), (2.7) with (2.9), we have

$$(2.10)\qquad \{r\}+\{1-r\}=-2\{2\}=-2\{-1\}=2\{1/2\},\quad 0<r<1.$$

Setting $r=(1+s)^{-1}$, $s>0$, (2.10) is equivalent with

$$(2.11)\qquad \{1+s\}-\{1+s^{-1}\}=2\{2\}=2\{-1\},\quad s>0.$$

Using (2.7) and (2.11), we obtain

$$\{-t\}+\{-t^{-1}\}=2\{-1\}=2\{2\}, \quad t>0. \tag{2.12}$$

If we set $r_i=r^i$, $r>1$, then (2.4) gives

$$\{r^2\}=2\{r\}+\{1+r^{-1}\}-\{1+r\}=2\{r\}+2\{-r\}-2\{-1\}, \quad r>1. \tag{2.13}$$

Actually, (2.13) is valid for r in $\mathbb{R}^\times$ through (2.6), (2.12) and symmetry with respect to -1. We note that it is valid for $r=1$ through (2.5).

2.14. Proposition. *$H_3(W/G^\delta)$ is 2-divisible. Moreover, there is a surjective homomorphism*

$$L: H_3(W/G^\delta)\to\mathbb{R}$$

with the properties:

(a) *L is a bijective map between the set of generators $\{r\}$, $r>1$ and the open interval $(0,\pi^2/6)$.*

(b) *As a function of r, $r>1$, $L\{r\}$ is strictly increasing and analytic in r.*

(c) *Setting $x=1-r^{-1}$ so that $r=(1-x)^{-1}$, $r>1$ and $0<x<1$, then L can be defined through the Rogers' L-function f given by the relations:*

$$L\{r\}=f(x)=f(1-r^{-1}), \quad r>1$$

and

$$f(x)=\sum_{n\geq 1} x^n/n^2+(\log x)\cdot(\log(1-x))/2, \quad 0<x<1.$$

Proof. The first assertion follows from (2.13) together with the fact that $H_3(W/G^\delta)$ is generated by the symbols $\{s\}$, $s>1$. We only have to set $s=r^2$, $r>1$ and apply (2.13).

Rogers' L-function f was studied in [12]. It is clearly analytic for $0<x<1$ and can be defined by the improper integral:

$$f(x)=-\frac{1}{2}\int_0^x\left(\frac{\log(1-t)}{t}+\frac{\log t}{1-t}\right)\mathrm{d}t, \quad 0<x<1.$$

It is evidently positive and strictly increasing with $\lim_{x\to 1} f(x)=\pi^2/6$. Moreover, Rogers showed that f satisfies the identities:

$$f(x)+f(1-x)=\pi^2/6, \quad 0<x<1,$$

$$f(x)+f(y)=f(xy)+f\left(\frac{x-xy}{1-xy}\right)+f\left(\frac{y-xy}{1-xy}\right), \quad 0<x,\ y<1.$$

Using the first of these, the second one can be rewritten in the form:

$$f\left(\frac{1-x}{1-xy}\right)-f\left(\frac{y-xy}{1-xy}\right)+f(y)-f(xy)+f(x)=\pi^2/6. \tag{2.15}$$

Using (2.6), the defining relation (2.4) can be translated to the defining relation in terms of generators $\{s\}$, $0<s<1$, satisfying

$$(2.16)\qquad \begin{aligned}&\{s_1\}-\{s_2\}+\{s_2/s_1\}-\{(1-s_1^{-1})/(1-s_2^{-1})\}+\{(1-s_1)/(1-s_2)\}=0,\\ &0<s_2<s_1<1, \qquad s_i=r_i^{-1} \text{ in } (2.4).\end{aligned}$$

Setting $s_1=(1-x)/(1-xy)$ and $s_2=(y-xy)/(1-xy)$ in (2.16), we obtain

$$(2.17)\qquad \left\{\frac{1-x}{1-xy}\right\}-\left\{\frac{y-xy}{1-xy}\right\}+\{y\}-\{xy\}+\{x\}=0.$$

Evidently, $y=s_2/s_1$ and $x=(1-s_1)/(1-s_2)$ and a comparison of (2.15) and (2.17) shows that we have a homomorphism $L:H_3(W/G^\delta)\to\mathbb{R}$ such that:

$$L\{s\}=f(s)-\pi^2/6=-f(1-s), \quad 0<s<1.$$

We get the desired conclusions through the functional equation (2.6). $\square$

We will now relate these results to the homology of $\mathrm{SL}(2,\mathbb{R})$. From Proposition 1.9, the chain complex $C_*(W)$ has the integral homology of a circle. We therefore have the exact sequences (set $C_i(W)=C_i$):

$$(2.18)\qquad 0\to Z_1\to C_1\to C_0\to\mathbb{Z}\to 0,$$

$$(2.19)\qquad 0\to B_1\to Z_1\to\mathbb{Z}\to 0,$$

$$(2.20)\qquad \cdots\to C_{n+2}\to C_{n+1}\to\cdots\to C_3\to C_2\to B_1\to 0.$$

We note that (2.20) is a $\mathbb{Z}\mathrm{SL}(2,\mathbb{R})$-free resolution of B_1. It follows that:

$$(2.21)\qquad H_0(\mathrm{SL}(2,\mathbb{R}),B_1)\cong\mathbb{Z} \quad\text{and}\quad H_i(\mathrm{SL}(2,\mathbb{R}),B_1)\cong H_{i+2}(W/G^\delta), \quad i>0.$$

We can use (2.18) to get a spectral sequence converging to the homology of $\mathrm{SL}(2,\mathbb{R})$. The $''E^1_{i,j}$-terms are all 0 for $j>2$ and we have the rest below:

$$(2.22)\qquad \begin{array}{ccccc} Z_1\otimes_{\mathbb{Z}G}\mathbb{Z} & H_1(G,Z_1) & H_2(G,Z_1) & \cdots & H_i(G,Z_1)\\ \mathbb{Z} & \Lambda^1_{\mathbb{Z}}(A) & \Lambda^2_{\mathbb{Z}}(A) & \cdots & \Lambda^i_{\mathbb{Z}}(A)\\ \mathbb{Z} & \Lambda^1_{\mathbb{Z}}(A) & \Lambda^2_{\mathbb{Z}}(A) & \cdots & \Lambda^i_{\mathbb{Z}}(A)\end{array}\ \Big\downarrow d^1$$

Just as in (2.1), $d^1:''E^1_{i,1}\to''E^1_{i,0}$ is $(-1)^i-1$, $i\geq 0$. Thus, it is alternately zero and bijective because $\Lambda^i_{\mathbb{Z}}(A)$ is a $\mathbb{Q}$-vector space for $i>0$. (2.19) yields the long homology exact sequence:

$$(2.23)\qquad \begin{aligned}&\cdots\to H_i(\mathrm{SL}(2,\mathbb{R}),B_1)\to H_i(\mathrm{SL}(2,\mathbb{R}),Z_1)\to H_i(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})\\ &\to H_{i-1}(\mathrm{SL}(2,\mathbb{R}),B_1)\to\ldots, \quad i>0.\end{aligned}$$

Since $H_1(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})=0$, we can conclude from (2.23) and (2.21) that:

$$(2.24)\qquad Z_1\otimes_{\mathbb{Z}G}\mathbb{Z}\cong\mathbb{Z}^2 \quad\text{and}\quad d^1:Z_1\otimes_{\mathbb{Z}G}\mathbb{Z}\to\mathbb{Z} \text{ is surjective.}$$

The exact sequence (2.19) arose from the 1-dimensional homology of the circle. We can split (2.19) as free abelian groups by the choice of a fundamental cycle:

$$(2.25)\qquad [S(\mathbb{R}^2)] = \sum_{0\le i\le 3} w^i\cdot(-\infty,0),\qquad w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We note that a column vector is written as $\binom{y}{x}$ so that the Weyl group element w corresponds to rotation through $\pi/2$ radians in the counter-clockwise direction. w inverts the subgroup A. We note that:

(2.26) A fixes the fundamental 1-cycle $[S(\mathbb{R}^2)]$.

The map $d^1: H_i(G,Z_1)\to H_i(G,C_1)\cong \Lambda^i_{\mathbb{Z}}(A)\cdot(-\infty,0)$ is induced by the inclusion map on the coefficient groups. If $c\in\Lambda^i_{\mathbb{Z}}(A)\cong H_i(A,\mathbb{Z})$, then the inclusion of A into G together with (2.25) and (2.26) shows that we have

$$(2.27)\qquad \iota: H_i(A,Z_1)\to H_i(G,Z_1),\qquad H_i(A,Z_1)\cong H_i(A,B_1)\amalg H_i(A,\mathbb{Z}).$$

$d^1\circ\iota$ carries $c\otimes[S(\mathbb{R}^2)]$ onto $2\cdot((-1)^i+1)\cdot c\otimes(-\infty,0)$. This shows:

(2.28) $d^1: H_i(G,Z_1)\to\Lambda^i_{\mathbb{Z}}(A)$ is surjective for i even, zero for i odd.

We can display the $''E^2_{i,j}$ terms:

$$(2.29)\qquad \begin{array}{cccccccc} \mathbb{Z} & H_1(G,Z_1) & \ker_2 & H_3(G,Z_1) & \ker_4 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \big\downarrow d^2 \\ \mathbb{Z} & 0 & \Lambda^2_{\mathbb{Z}}(A) & 0 & \Lambda^4_{\mathbb{Z}}(A) & \cdots & \end{array}$$

Here $\ker_{2j}$ denotes the kernel of d^1 in (2.28).

The map $d^2: H_1(G,Z_1)\to\Lambda^2_{\mathbb{Z}}(A)$ can be described. We compose it with the surjective map $\eta: H_1(G,B_1)\to H_1(G,Z_1)$ in (2.23). Using the isomorphism in (2.21), $d^2\circ\eta$ is simply d^3 in (2.2). The map d^3 can be determined as in Dupont–Sah [4]. The necessary modifications are compiled in Appendix A. We merely summarize the results:

$$(2.30)\qquad \begin{array}{ccc} H_1(G,B_1) & \cong & H_3(W/G^\delta)\\ \Big\downarrow{\scriptstyle\eta} & & \Big\downarrow{\scriptstyle d^3}\\ H_1(G,Z_1) & \xrightarrow{\;d^2\;} & \Lambda^2_{\mathbb{Z}}(A) \end{array}\qquad d^3\{r\}=2\cdot(r\wedge(r-1)),\quad r>1.$$

Since $\Lambda^2_{\mathbb{Z}}(A)$ is a $\mathbb{Q}$-vector space and $H_3(W/G^\delta)$ is 2-divisible, the factor of 2 has no effect on the image of d^2 or d^3. Since $H_3(W/G^\delta)$ is conjecturally a $\mathbb{Q}$-vector space, there is presumably no problem with $\ker d^3$ if the factor of 2 is ignored.

In the complex setting, d^3 was denoted by λ. The formula was first found by Bloch–Wigner (without the factor of 2) using a slightly different chain complex for PSL(2, ℂ), see Dupont–Sah [4]. Indeed, a portion of the argument used in Dupont–Sah [4] involved showing that the alternation rules (2.5) through (2.7) were consequences of (2.4) in the complex case (where r is allowed to be any complex number other than 0 or 1). In the present case, we took advantage of the ordering in $\mathbb{R}$ and started with an *alternating* chain complex.

We now read off (2.29) the exact sequences, $G = \mathrm{SL}(2,\mathbb{R})$:

(2.31) $$0 \to H_3(G,\mathbb{Z}) \to H_1(G,Z_1) \to \Lambda^2_{\mathbb{Z}}(A) \to H_2(G,\mathbb{Z}) \to \mathbb{Z} \to 0,$$

(2.32) $$0 \to H_{2i+3}(G,\mathbb{Z}) \to H_{2i+1}(G,Z_1) \to \Lambda^{2i+2}_{\mathbb{Z}}(A) \to H_{2i+2}(G,\mathbb{Z}) \to \ker_{2i} \to 0,$$ $i \geq 1$.

We can combine (2.31) with part of (2.23):

(2.33)
$$\begin{tikzcd}
 & & H_2(G,\mathbb{Z}) \cong K_2^0(\mathbb{R}) \amalg \mathbb{Z}\cdot c(-1,-1) \arrow[d] & & & \\
 & & H_1(G,B_1) \cong H_3(W/G^\delta) \arrow[d, "\eta"] & & & \\
0 \arrow[r] & H_3(G,\mathbb{Z}) \arrow[r] & H_1(G,Z_1) \arrow[r, "d^2"] \arrow[d] & \Lambda^2_{\mathbb{Z}}(A) \arrow[r, "\mathrm{sym}"] & H_2(G,\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & 0 & & &
\end{tikzcd}$$

$K_2^0(\mathbb{R})$ is just the image of $\Lambda^2_{\mathbb{Z}}(A)$ under the 'K_2-symbol' map. It arises from the inclusion of A into G. An examination of the spectral sequence terms displayed in (2.2) shows the known result that $K_2^0(\mathbb{R})$ is just $H_2(\tilde{G},\mathbb{Z})$. In fact, we have the following exact sequence:

(2.34)
$$\xrightarrow{d^3} \Lambda^3_{\mathbb{Z}}(\mathbb{R}) \to H_3(\tilde{G},\mathbb{Z}) \to H_3(W/G^\delta)$$
$$\xrightarrow{d^3} \Lambda^2_{\mathbb{Z}}(\mathbb{R}) \xrightarrow{\mathrm{sym}} H_2(\tilde{G},\mathbb{Z}) \cong K_2^0(\mathbb{R}) \to 0.$$

The argument used in deriving (2.28) and (2.29) also shows that the vertical map from $H_2(G,\mathbb{Z})$ down to $H_1(G,B_1)$ in (2.33) is zero on $K_2^0(\mathbb{R})$. The image of $c(-1,-1)$ in $H_1(G,B_1) \cong H_3(W/G^\delta)$ can be explicitly determined, see Theorem 3.24, and the homomorphism L of Proposition 2.14 can be used to show that $\mathbb{Z}\cdot c(-1,-1)$ is mapped into $H_1(G,B_1)$ injectively. (2.30) shows that the image of $c(-1,-1)$ in $H_1(G,B_1)$ lies in the kernel of λ. We summarize this with the following:

2.35. Theorem. *We have the following diagram of maps with exact row and column:*

(2.36)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & & & \\
 & & \mathbb{Z}=\mathbb{Z}\cdot c(-1,-1) \arrow[d] & & & & \\
 & & H_1(G,B_1) \arrow[r, phantom, "\cong"] \arrow[d, "\eta"] & H_3(W/G^\delta) \arrow[d, "d^3"] & & & \\
0 \arrow[r] & H_3(G,\mathbb{Z}) \arrow[r] & H_1(G,Z_1) \arrow[r, "d^2"] \arrow[d] & \Lambda^2_{\mathbb{Z}}(A) \arrow[r, "\mathrm{sym}"] & H_2(G,\mathbb{Z}) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & 0 & & & &
\end{tikzcd}$$

Except for the explicit statement in Theorem 3.24 that the ℝ-valued homomorphism L on $H_1(G,B_1)\cong H_3(W/G^\delta)$ is definitely nonzero on the image of $c(-1,-1)$, a more conceptual argument can be given as follows.

We begin with the Hochschild-Serre spectral sequence associated to (1.1). Its ${}'E^2_{i,j}$-terms are displayed below:

(2.37)
$$\begin{array}{lll}
0\cdots & & \xleftarrow{\;d^2\;} \\
\mathbb{Z} & 0 & H_2(G,H_1(\pi_1(G),\mathbb{Z})),\ldots,H_i(G,H_1(\pi_1(G),\mathbb{Z})),\ldots \\
\mathbb{Z} & 0 & H_2(G,H_0(\pi_1(G),\mathbb{Z})),\ldots,H_i(G,H_0(\pi_1(G),\mathbb{Z})),\ldots
\end{array}$$

We note that $H_j(\pi_1(G),\mathbb{Z})\cong\mathbb{Z}\cong\pi_1(G)$, $j=0,1$ and $H_j(\pi_1(G),\mathbb{Z})=0$ for $j>1$. The spectral sequence simply exhibits the fact that $K(\tilde{G},1)$ is homotopically a circle bundle over $K(G,1)$.

Since $H_1(\tilde{G},\mathbb{Z})=0$, we read off the exact sequences from (2.37):

(2.38) $$0\to H_2(\tilde{G},\mathbb{Z})\to H_2(G,\mathbb{Z})\to\mathbb{Z}\to 0,$$

(2.39) $$H_4(G,\mathbb{Z})\xrightarrow{\;d^2\;}H_2(G,H_1(\pi_1(G),\mathbb{Z}))\to H_3(\tilde{G},\mathbb{Z})\to H_3(G,\mathbb{Z})\to 0.$$

We recover from (2.38) the well-known result that $H_2(\tilde{G},\mathbb{Z})$ is just the ℚ-vector space $K_2^0(\mathbb{R})$. We assert that:

(2.40) the map d^2 in (2.39) is the zero map.

Up to factors of 2-power, we can deduce the result from Milnor [11] on the level of cohomology. Namely, the exact sequence (1.1) corresponds to an element $c \in H^2(G, \mathbb{Z})$. If we let $PG = \mathrm{PSL}(2, \mathbb{R})$, then we have the exact sequence:

$$(2.41) \qquad 1 \to \pm I \to G \to PG \to 1.$$

From Hochschild–Serre homology and cohomology spectral sequences, we have the exact sequences:

$$(2.42) \qquad 0 \to H_2(G, \mathbb{Z}) \to H_2(PG, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

$$(2.43) \qquad 0 \to H^2(PG, \mathbb{Z}) \cong \mathbb{Z} \to H^2(G, \mathbb{Z}) \cong \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Actually, (2.43) follows from universal coefficient theorem together with the known structure of the terms in (2.42). It follows that a generator e of $H^2(PG, \mathbb{Z})$ can be taken to be $2c$ because c generates $H^2(G, \mathbb{Z})$. We note that c takes the value 0 on $K_2^0(\mathbb{R})$ and the value 1 on $c(-1, -1)$.

A general theorem of Hochschild–Serre [6] shows that d_2 in the cohomology spectral sequence associated to (1.1) is cup-product with c. As a consequence, (2.40) would follow from the vanishing of the square of c. This is the case as stated in Milnor [11]. Actually, the assertion was made for e, but there is no difficulty extending the argument to c.

We can also give the argument directly. Namely, it is enough to show that

$$H_2(G, H_1(\pi_1(G), \mathbb{Z})) \cong H_2(G, \mathbb{Z}) \cong K_2^0(\mathbb{R}) \amalg \mathbb{Z} \cdot c(-1, -1)$$

survives to $'E^\infty$. First recall that the part corresponding to $K_2^0(\mathbb{R})$ is the image of $H_2(A, \mathbb{Z})$ under the inclusion map of A into G. Now $H_2(A, \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}^2(A)$ and is generated by mapping the free abelian group $\mathbb{Z}^2$ into A. This corresponds to mapping a 2-torus into $B(G, 1)$ through $B(A, 1)$. Since the exact sequence (1.1) splits on the subgroup A, any such homomorphism of $\mathbb{Z}^2$ into A can be viewed as the restriction of a homomorphism of $\mathbb{Z} \times \mathbb{Z}^2$ into $\mathbb{Z} \times A$ to the second factor with the first factor equal to the identity map of $\mathbb{Z}$. This translates into the statement that the $K_2^0(\mathbb{R})$ part of $H_2(G, H_1(\pi_1(G), \mathbb{Z}))$ lifts to elements of $H_3(\tilde{G}, \mathbb{Z})$. We next prove that $c(-1, -1)$ also survives to $'E^\infty$. For this, we follow Milnor [9]. Consider a compact Riemann surface $\mathcal{M}$ of genus $g > 1$. Its fundamental group $\pi_1(\mathcal{M})$ can therefore be faithfully realized (in many ways) in PG as a cocompact discrete group. $\pi_1(\mathcal{M}) \backslash PG$ can then be identified with the canonical unit tangent bundle over $\mathcal{M}$. The fundamental group of this unit tangent bundle then satisfies the central exact sequence:

$$(2.44) \qquad 0 \to \mathbb{Z} \to \tilde{\pi}_1(\mathcal{M}) \to \pi_1(\mathcal{M}) \to 1.$$

If a_i, b_i, $1 \le i \le g$, are the usual generators of $\pi_1(\mathcal{M})$ with the single defining relation: $\prod_i [a_i, b_i] = 1$, then the generator of $\mathbb{Z}$ in (2.44) can be taken to be $\prod_i [\tilde{a}_i, \tilde{b}_i]$ where $\tilde{a}_i, \tilde{b}_i$ are arbitrary representatives of a_i, b_i respectively. The realization of $\pi_1(\mathcal{M})$ in PG then lifts to a homomorphism of $\tilde{\pi}_1(\mathcal{M})$ into $\tilde{G}$ and the generator of

$\mathbb{Z}$ in (2.44) is then mapped onto the $(2-2g)$-th power of the generator of $\pi_1(PG)$, or to the $(1-g)$-th multiple of the element denoted by $c(-1,-1)$. This group homomorphism corresponds to a homotopy class of maps of the unit tangent bundle of $\mathscr{M}$ into $B(\tilde{G}, 1)$. We have the induced map on the third integral homology. We get a lift of $c(-1,-1)$ when $\mathscr{M}$ is chosen to have genus 2. (The gist of this argument was worked out during a conversation with A.T. Vasquez.)

With (2.40) at hand, we can fit (2.39) into (2.36) using (2.34). We need to check the commutativity of various maps. This will be left to the careful reader.

As shown in Theorem 3.24, there are homomorphisms:

$$(2.45) \qquad \begin{aligned} &\lambda : H_1(G, B_1) \cong H_3(W/G^\delta) \to \Lambda^2_{\mathbb{Z}}(\mathbb{R}^+), \qquad \lambda\{r\} = r \wedge (r-1), \quad r>1. \\ &L : H_1(G, B_1) \cong H_3(W/G^\delta) \to \mathbb{R}, \qquad L \text{ as in Proposition 2.14.} \end{aligned}$$

λ is 0 on the image of $c(-1,-1)$ and is just d^2 in (2.36) up to a factor a power of 2. Thus λ is 0 on the image of $H_3(G, \mathbb{Z})$ when we factor it through $H_1(G, Z_1)$. We can view λ as the analogue of the Dehn invariant, see Dupont–Sah [4]. L takes on the value $\pm 2 \cdot 2\pi^2$ on the image of $c(-1,-1)$. The ambiguity of ± 1 comes about because we have not kept tract of the sign in tracking down our higher differential. The first factor of 2 can be seen from the fact that SL(2, ℝ) doubly covers PSL(2, ℝ), compare (3.17). The constant $2\pi^2$ is the 3-dimensional volume of the 3-sphere of radius 1. We can therefore view L as the analogue of the volume invariant in hyperbolic 3-space, see Dupont–Sah [4]. In analogy with the yet to be resolved Hilbert third problem in hyperbolic space, it appears reasonable to ask:

(2.46) Do λ and L separate the points of $H_3(W/G^\delta)$?

The conjectured $\mathbb{Q}$-vector space structure on $H_3(W/G^\delta)$ would follow from (2.46) together with p-divisibility of $H_3(W/G^\delta)$ for all odd primes p. We note that $H_3(G, \mathbb{Z})$ is known to have a $\mathbb{Q}/\mathbb{Z}$ direct summand coming from the inclusion of finite cyclic groups into the subgroup SO(2, ℝ), compare Dupont–Sah [4]. Diagram (2.36) clearly points out the compatibility of the conjecture concerning the structure of $H_3(G, \mathbb{Z})$. Since $\lambda\{2\} = 0$ and d^2 has kernel $H_3(\mathrm{SL}(2, \mathbb{R}), \mathbb{Z})$, (2.46) is equivalent with (compare (3.37)):

(2.47) Is $L_{\mathrm{red}} : H_3(\mathrm{SL}(2, \mathbb{R}), \mathbb{Z}) \to \mathbb{R}/4\pi^2\mathbb{Z}$ injective?

3. Determination of the image of $c(-1,-1)$

We begin with the review of known results that are more or less scattered and usually presented in somewhat different forms.

Consider a presentation of an abstract group Γ in the form of an exact sequence:

$$(3.1) \qquad 1 \to R \to F(S) \xrightarrow{\gamma} \Gamma \to 1.$$

Here $F(S)$ is the free group based on the set S. Given any element in $F(S)$:

(3.2) $$w=\prod_i s(i)^{\varepsilon(i)}, \qquad \varepsilon(i)=\pm 1 \quad \text{and} \quad s(i)\in S, \quad 1\le i\le n,$$

we define the exponential sum with respect to $s\in S$ by the formula:

(3.3) $$e_s(w)=\sum_{s(i)=s}\varepsilon(i).$$

We also define $p_i(w)$ by the formula:

(3.4) $$p_i(w)=\prod_{j\le i} s(j)^{\varepsilon(j)}, \qquad 1\le j\le n, \qquad p_0(w)=1, \quad p_n(w)=w.$$

3.5. Proposition. *Suppose that the element w given in* (3.2) *has zero exponential sum with respect to each s in S and that $\gamma(w)=1$. Equivalently, $w\in R\cap[F(S),F(S)]$. Set $\gamma_i=\gamma(p_i(w))$. Then,*

(a) *The 2-chain:*

$$c(w)=\sum_{1\le i\le n}\varepsilon(i)[\gamma_{i-(\varepsilon(i)+1)/2}\,|\,\gamma(s(i))]$$

represents a 2-cycle in the nonhomogeneous bar formulation of the Eilenberg-MacLane space $B\Gamma=B(\Gamma,1)$.

(b) *When $\Gamma=\Gamma_g$, $g\ge 1$, is the fundamental group of a compact surface of genus g with generators a_i,b_i, $1\le i\le g$, and the single defining relation $w=\prod_i[a_i,b_i]$ (i.e., R is the normal subgroup generated by w), then $c(w)$ represents a generator of $H_2(\Gamma_g,\mathbb{Z})\cong\mathbb{Z}$.*

Remark. $c(w)$ is invariant under insertion of $a\cdot a^{-1}$ and changes by a boundary under conjugation of w.

Proof. The boundary of $c(w)$ is just the element:

$$\sum_{1\le i\le n}\varepsilon(i)[\gamma(s(i))]-\sum_{1\le i\le n}[\gamma_i]+\sum_{1\le i\le n}[\gamma_{i-1}].$$

It is therefore 0 from the assumption on exponential sum. The assumption that $\gamma(w)=1$ is needed to obtain the precise form of the last two sums in order to get cancellation. We note that $\gamma_0=\gamma_n=1$ so that the terms corresponding to $i=0$ and n may become 0 independently through the normalization of the nonhomogeneous formulation. We therefore have (a).

The proof of (b) is essentially contained in Milnor [9]. When $g=1$,

$$c(w)=[\gamma(a)\,|\,\gamma(b)]-[\gamma(b)\,|\,\gamma(a)]$$

is well-known to represent a generator of $H_2(\mathbb{Z}^2,\mathbb{Z})$, compare Sah [13]. For $g>1$, we may geometrically realize Γ_g as the fundamental group of a compact Riemann surface of genus g, for example, a hyperelliptic curve defined by

$$Y^2=\prod_{1\le i\le 2g+2}(X-i).$$

The 2-chain can be seen to be a decomposition of the classical $4g$-sided fundamental polygon in the hyperbolic plane. More precisely, the Riemann surface is a model of $K(\Gamma_g, 1)$ with universal covering space equal to the hyperbolic plane. The 2-chain $c(w)$ corresponding to the defining relation exhibited projects bijectively to a cell decomposition of the fundamental 2-cycle on the Riemann surface. Since $c(w)$ is a 2-cycle in $H_2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$, it must correspond to a generator. We note that the identification of $H_2(\Gamma_g, \mathbb{Z})$ with $\mathbb{Z}$ in the present argument used the uniformization theorem in complex analysis and basic facts of group homology. □

The preceding argument can be reorganized in the following form: Let us form the following exact sequence of groups:

$$1 \to \mathbb{Z} = \langle z \rangle \to \tilde{\Gamma}_g \to \Gamma_g \to 1. \tag{3.6}$$

$\tilde{\Gamma}_g$ is defined by generators $\tilde{a}_i, \tilde{b}_i$, $1 \le i \le g$ and z with defining relations:

$$z = \prod_i [\tilde{a}_i, \tilde{b}_i] \quad \text{and} \quad [z, \tilde{a}_i] = 1 = [z, \tilde{b}_i], \quad 1 \le i \le g.$$

Since (3.6) is now a central extension, the element z is independent of the choice of the representatives $\tilde{a}_i, \tilde{b}_i$ chosen in $\tilde{\Gamma}_g$ covering the generators a_i, b_i of Γ_g. $\tilde{\Gamma}_1$ is usually called the Heisenberg group. In $\tilde{G}$, let the center be generated by t so that $t^2 = c(-1, -1)$ in the multiplicative notation. For any homomorphism f of Γ_g into PSL(2, ℝ), we can define a homomorphism $\tilde{f}$ of $\tilde{\Gamma}_g$ into $\tilde{G}$ by mapping the generators $\tilde{a}_i, \tilde{b}_i$ into arbitrary representatives of $f(a_i), f(b_i)$ in $\tilde{G}$. The element $\tilde{f}(z)$ must then be a power $t^{c(f)}$ of t. The exponent $c(f)$ depends only on f and will be called the Chern number of f. When Γ_g, $g > 1$, is mapped by f onto the fundamental group of a compact Riemann surface of genus g (realized as a conjugacy class of torsionfree Fuchsian groups of the first kind), then the results in Milnor [9] show that $c(f) = 2 - 2g$ is simply the Euler characteristic of the Riemann surface. In general, $c(f)$ is the first Chern number of the flat 2-plane bundle defined by f.

Remark. In passing, it should be noted that extensions of the preceding result have appeared in many, many disguises. An amusing historical account can be found in a forthcoming paper of Kra [7].

We now proceed to the determination of the image of $c(-1, -1)$ in $H_1(G, B_\cdot) \cong H_3(W/G^\delta)$. We first describe $\tilde{G}$ following Milnor [10], or Steinberg [17]. The 1-parameter group $e_{ij}(t)$, $t \in \mathbb{R}$, in SL(2, ℝ), $(i, j) = (1, 2)$ or $(2, 1)$, is covered by the 1-parameter group $x_{ij}(t)$ in $\tilde{G}$. Since ℝ is divisible and $\pi_1(G) \cong \mathbb{Z}$, these liftings are absolutely unique. According to Milnor [10, §10], $c(-1, -1)$ is just $(x_{12} x_{21}^{-1} x_{12})^4$ with $x_{ij} = x_{ij}(1)$. We note that $x_{12} \cdot x_{21}^{-1} \cdot x_{12}$ has image equal to the Weyl element w in SL(2, ℝ):

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since $c(-1,-1)$ lies in the center of $\tilde{G}$, we can conjugate $x_{12}x_{21}^{-1}x_{12}$ at will in $\tilde{G}$. We let $h(a)$, $a\in\mathbb{R}^+$, be the 1-parameter subgroup of G covering the diagonal subgroup A of $\mathrm{SL}(2,\mathbb{R})$ so that $h(a)$ maps onto the diagonal matrix

$$\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix},\qquad ab=1,\quad a,b>0.$$

Again the divisibility of A shows that this lift is absolutely unique. We now note the identities in $\mathrm{SL}(2,\mathbb{R})$:

$$\begin{aligned}&[e_{21}(s),h(a)]=e_{21}(s\cdot(1-a^{-2})), &&s\text{ in }\mathbb{R},\quad a\text{ in }\mathbb{R}^+,\\ &[h(a),e_{12}(t)]=e_{12}(t\cdot(a^2-1)), &&t\text{ in }\mathbb{R},\quad a\text{ in }\mathbb{R}^+.\end{aligned}\tag{3.7}$$

It follows that the same identities are valid in $\tilde{G}$ when e_{ij} is replaced by x_{ij}. Notice that we have used the fact that $\mathbb{Z}$ contains only the trivial divisible subgroup in an essential way. Direct consequence of (3.7) is:

$$\begin{aligned}e_{21}^{-1}e_{12}^{2}&=[e_{21}(-2),h(2^{1/2})]\cdot[h(2^{1/2}),e_{12}(2)]\\ &=e_{21}^{-2}\cdot[h(2^{1/2}),e_{21}^{2}e_{12}^{2}]\cdot e_{21}^{2}\\ &=e_{12}^{2}\cdot[e_{12}^{-2}e_{21}^{-2},h(2^{1/2})]\cdot e_{12}^{-2},\qquad e_{ij}=e_{ij}(1).\end{aligned}\tag{3.8}$$

$$e_{21}^{2}\cdot e_{12}^{2}=\begin{pmatrix}1&2\\2&5\end{pmatrix}\quad\text{in }\mathrm{SL}(2,\mathbb{R}).$$

We can therefore conclude that:

$$c(-1,-1)=[h(2^{1/2}),x_{21}^{2}x_{12}^{2}]^4=[x_{12}^{-2}x_{21}^{-2},h(2^{1/2})]^4\quad\text{in }\tilde{G}\tag{3.9}$$

This means that we can map Γ_4 into $\tilde{G}$ so that a_i,b_i are mapped respectively onto $h(2^{1/2})$, $e_{21}^{2}\cdot e_{12}^{2}$, $1\le i\le 4$, and with a generator of $H_2(\Gamma_4,\mathbb{Z})$ mapped onto $c(-1,-1)$.

We now go back to the vertical part of (2.33). We need to find a 2-chain of G with coefficient in Z_1 so that it projects onto a 2-cycle representing $c(-1,-1)$. We use the idea connected with (2.25). Set:

$$w_0=\left[h(2^{1/2}),\begin{pmatrix}1&2\\2&5\end{pmatrix}\right],\qquad w_0\text{ conjugate to }w\text{ in }\mathrm{SL}(2,\mathbb{R}).\tag{3.10}$$

It follows that w_0 has order 4 and (2.19) can be split as abelian groups by using the following fundamental 1-cycle:

$$\sum_{0\le i\le 3} w_0^i(r,s),\qquad s=w_0(r),\quad -\infty\le r<\infty\tag{3.11}$$

What follows is the standard 'staircase' argument in tracing down higher differentials in a spectral sequence associated to a double complex.

The relevant 2-chain of (G,Z_1) projecting onto a 2-cycle of $(G,\mathbb{Z})$ representing the same class as $c(-1,-1)$ can be taken to be:

$$X=\sum_i\left([w_0^ix\,|\,y]-[w_0^iy\,|\,x]+[w_0^i\,|\,x]-[w_0^i\,|\,y]\right)\otimes\sum_i w_0^i(r,s),\tag{3.12}$$
$$0\le i\le 3,\quad x=h(2^{1/2}),\quad y=\begin{pmatrix}1&2\\2&5\end{pmatrix}.$$

The first group of 16 terms in (3.12) comes from (a) of Proposition 3.5 by using the map described after (3.9). Using $xy=w_0yx$, $w_0^4=1$, $(\partial_G\otimes 1)\cdot X$ is easily computed and we obtain

$$4[x]\otimes(y^{-1}-1)\cdot\sum_i w_0^i(r,s)-4[y]\otimes(x^{-1}-1)\cdot\sum_i w_0^i(r,s).\tag{3.13}$$

We remind ourselves that the tensor product in (3.13) is over $\mathbb{Z}G$. As a direct check, using $x^{-1}y^{-1}=y^{-1}x^{-1}w_0$, $w_0^4=1$, (3.13) is a 1-cycle of G with coefficient in Z_1. However, G acts trivially on $\mathbb{Z}\cong Z_1/B_1$, thus the second factors in the two terms of (3.13) are actually in B_1 so that (3.13) is really a 1-cycle of (G,B_1). Note that we are free to choose the ray r to simplify our task.

In order to keep track of the steps, we assume that 2-chains in $C_2(W)$, say $Y(y)$ and $Y(x)$, have been found so that ∂_W maps them onto $(y^{-1}-1)\cdot\sum_i w_0^i(r,s)$ and $(x^{-1}-1)\cdot\sum_i w_0^i(r,s)$ respectively. We then apply $\partial_G\otimes 1$ to $4[x]\otimes Y(y)-4[y]\otimes Y(x)$ to get:

$$4\cdot Z,\quad Z=(x^{-1}-1)\cdot Y(y)-(y^{-1}-1)\cdot Y(x)\in C_2(W).\tag{3.14}$$

We are assured (and it can be checked directly) that $\partial_W Z=0$ so that the vanishing of the higher homology groups of W (above dimension 1) implies:

$$Z=\partial_W U,\quad U\in C_3(W).\tag{3.15}$$

The class of U in $H_3(W/G^\delta)$ is then the desired image of $c(-1,-1)$. We note that we have not bothered keeping track of signs of ± 1 so that a more precise statement is that the class of U in $H_3(W/G^\delta)$ is the image of $\pm c(-1,-1)$.

We now select r to be the ray $-\infty$. We will abbreviate the ray $\mathbb{R}^+\binom{y}{x}$ to y/x and note that $-y/-x$ is the ray y/x rotated through a half circle. It is then straightforward to check that the following choices will work:

$$\begin{aligned}Y(x)&=(-\infty,3/10,3/5)-(3/10,3/5,\infty)+(\infty,-3/-10,-3/-5)\\&\quad-(-3/-10,-3/-5,-\infty),\\ Y(y)&=(-\infty,-5/2,3/5)-(-5/2,3/5,5/-1)-(3/5,\infty,5/-1)\\&\quad+(\infty,5/-1,5/-2)+(\infty,5/-2,-3/5)-(5/-2,-3/-5,-5/-1)\\&\quad-(-3/-5,-\infty,-5/1)+(-\infty,-5/1,-5/2).\end{aligned}\tag{3.16}$$

We note that the first half of $Y(x)$ and $Y(y)$ are equal to the corresponding second half through 180° rotation (or application of $-\mathrm{Id}$ in $\mathrm{SL}(2,\mathbb{R})$).

There is no difficulty computing Z of (3.14) from (3.16). With a little bit of patience, U can be found in the form:

$$U = V + w_0^2 \cdot V, \qquad w_0^2 = -\mathrm{Id} \in \mathrm{SL}(2, \mathbb{R}) \quad \text{where}$$

$$\begin{aligned} V = &-(-\infty, -5/4, 3/10, 3/5) - (-\infty, -5/2, -5/4, 3/5) \\ &-(-5/4, 3/10, 5/-1, 5/-2) - (-5/4, 3/10, 3/5, 5/-1) \\ &-(-5/2, -5/4, 3/5, 5/-1) - (3/10, \infty, 5/-1, 5/-2) \\ &+(3/10, 3/5, \infty, 5/-1). \end{aligned} \tag{3.17}$$

We must now compute the cross-ratio symbols of the 3-cells appearing in (3.17). Since we are now dealing with $C_3(W/G^\delta)$, we only have to multiply these answers by 8 to get the image of $c(-1,-1)$. Note that 8 comes from the 4 in (3.14) and the 2 in the expression for U in terms of V.

The calculations are now straightforward. We have:

3.18. Proposition. *The image of $c(-1,-1) \in H_2(G, \mathbb{Z})$ in $H_1(G, B_1) \cong H_3(W/G^\delta)$ can be represented by $8 \cdot v$ where v is the class of the following sum of cross-ratio symbols (up to a factor of ± 1):*

$$\begin{aligned} &-\{37/31\} - \{2 \cdot 31/5^2\} - \{2^2 \cdot 3 \cdot 7/53\} - \{37 \cdot 53/3^2 \cdot 5^2\} \\ &-\{3 \cdot 31/37\} - \{53/2^2 \cdot 7\} + \{2^3 \cdot 7/53\}. \end{aligned}$$

The rather strange looking prime numbers appearing in Proposition 3.18 are not very significant. The chain V in (3.17) is not unique; only its class v in $H_3(W/G^\delta)$ is. Indeed, (2.4) can be used to simplify the expression in Proposition 3.18.

Using $r_1 = 37/31$, $r_2 = 3$, we have

$$-\{37/31\} - \{3 \cdot 31/37\} = -\{3\} - \{31/3\} + \{37/3^2\} \quad \text{in } H_3(W/G^\delta). \tag{3.19}$$

Using $r_1 = 2^3 \cdot 7/53$, $r_2 = 2^2 \cdot 3 \cdot 7/53$, we have

$$\{2^3 \cdot 7/53\} - \{2^2 \cdot 3 \cdot 7/53\} = -\{3/2\} + \{31/3\} - \{2 \cdot 31/3^2\} \quad \text{in } H_3(W/G^\delta). \tag{3.20}$$

Using $r_1 = 2 \cdot 31/53$, $r_2 = 2 \cdot 31/5^2$, we have

$$-\{37 \cdot 53/3^2 \cdot 5^2\} - \{2 \cdot 31/5^2\} = -\{37/3^2\} - \{53/5^2\} - \{2 \cdot 31/53\} \quad \text{in } H_3(W/G^\delta). \tag{3.21}$$

It follows that:

$$v = -\{3/1\} - \{3/2\} - \{2 \cdot 31/3^2\} - \{2 \cdot 31/53\} - \{53/2^2 \cdot 7\} - \{53/5^2\} \quad \text{in } H_3(W/G^\delta). \tag{3.22}$$

Using (2.6) and (2.10), we see that:

$$v = -6\{2\} \quad \text{in } H_3(W/G^\delta). \tag{3.23}$$

3.24. Theorem. *Let* $\lambda : H_3(W/G^\delta) \to \Lambda^2_{\mathbb{Z}}(\mathbb{R}^+)$ *denote the homomorphism with* $\lambda\{r\} = r \wedge (r-1)$, $r > 1$ *and let* $L : H_3(W/G^\delta) \to \mathbb{R}$ *denote the homomorphism defined in Proposition* 2.14. *Then the image of* $c(-1,-1)$ *in* $H_1(G, B_1) \cong H_3(W/G^\delta)$ *is in the kernel of* λ *and is mapped by* L *to* $48 \cdot \pi^2/12 = 2 \cdot 2\pi^2$. *In particular, the image of* $\mathbb{Z} \cdot c(-1,-1)$ *in* $H_1(G, B_1)$ *is the infinite cyclic subgroup generated by* $48 \cdot \{2\}$.

As mentioned before, $H_3(W/G^\delta)$ is conjecturally a $\mathbb{Q}$-vector space. With the determination of the image of $c(-1,-1)$ at hand, we will show that $H_3(W/G^\delta)$ contains a direct summand isomorphic to $\mathbb{Q}$ and this summand can be selected to contain the image of $c(-1,-1)$. The uniqueness of this summand would depend on the absence of torsion in $H_3(W/G^\delta)$ (actually, it depends only on the absence of torsion of the form $\mathbb{Q}/\mathbb{Z}$). We accomplish this by using the known torsion in $H_3(\mathrm{SL}(2,\mathbb{R}), \mathbb{Z})$ arising from the finite subgroups of $\mathrm{SL}(2,\mathbb{R})$. It is evident that all such finite subgroups of $\mathrm{SL}(2,\mathbb{R})$ are conjugate to subgroups of $\mathrm{SO}(2,\mathbb{R})$, hence must be cyclic.

The following result is known but does not seem to be recorded in any readily available reference:

3.25. Proposition. *Let* Z_n *be a finite cyclic group generated by* g. *For any integer* $j \geq 0$, $H_{2j+1}(Z_n, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ *and a generator can be chosen to be the class represented by the following* $(2j+1)$*-cycle:*

$$\sum [g \mid g^{i(1)} \mid g \mid \cdots \mid g^{i(j)} \mid g], \quad 0 \leq i(s) \leq n-1, \quad 1 \leq s \leq j. \tag{3.26}$$

Proof. We know that $H_{2j+1}(Z_n, \mathbb{Z}) \cong H_{2j+2}(Z_n, \mathbb{Z}/n\mathbb{Z})$ via the connecting homomorphism associated to the coefficient sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0. \tag{3.27}$$

In particular, (3.26) arises from the following mod n $(2j+2)$-cycle:

$$\sum [g^{i(0)} \mid g \mid \cdots \mid g^{i(j)} \mid g], \quad 0 \leq i(s) \leq n-1, \quad 0 \leq s \leq j. \tag{3.28}$$

This sequence (3.27) corresponds to a 2-cocycle to $(Z_n, \mathbb{Z})$:

$$f(g^s, g^t) = \begin{cases} 0, & \text{if } 0 \leq s+t < n, \\ 1, & \text{if } n \leq s+t < 2n, \end{cases} \qquad 0 \leq s,\ t < n. \tag{3.29}$$

The cup product of $j+1$ factors of f then takes on the value 1 on the mod n $(2j+2)$-cycle in (3.28). This shows that the classes of (3.26), (3.28) and (3.29) are generators of order n in the respective groups. □

3.30. Proposition. *Let* g *be an element of finite order* $n > 2$ *in* $\mathrm{SL}(2,\mathbb{R})$. *The class of the generator from* (3.26) *is then mapped onto the class of the following* 1*-cycle of* (G, Z_1) *(compare* (2.25)*):*

$$c(g) = -[g] \otimes \sum_j g^j(r,s), \qquad 0 \leq j < n, \quad s = g(r), \quad -\infty \leq r < \infty. \tag{3.31}$$

If $g = h^m$ for some integer $m \geq 1$, then

(3.32) $\quad$ *$c(g)$ is homologous to $m \cdot c(h)$.*

Proof. We carry out the staircase argument in the spectral sequence (2.22) related to (2.18). In the double complex, the chain group in dimension 3 is

$$C_3(G) \otimes_{\mathbb{Z}G} C_0(W) \amalg C_2(G) \otimes_{\mathbb{Z}G} C_1(W) \amalg C_1(G) \otimes_{\mathbb{Z}G} Z_1.$$

We view the 3-cycle in (3.26) as a 3-cycle of $(G, \mathbb{Z})$ and cover it by the 3-chain $\sum_i [g \mid g^i \mid g] \otimes (r)$. Applying $\partial_G \otimes 1$ gives

$$\sum_i [g^i \mid g] \otimes (g^{-1} - 1)(r). \tag{3.33}$$

We need a 1-chain in $C_1(W)$ with boundary $(g^{-1} - 1)(r)$. Since $n > 2$, the 1-chain $(r, g^{-1}(r)) = -(g^{-1}(r), r)$ does the job. More generally, when $g = h^m$, we can use $\sum_j h^{-j}(r, h^{-1}(r))$, $0 \leq j < m$. We then apply $\partial_G \otimes 1$ again to the following element of $C_2(G) \otimes_{\mathbb{Z}G} C_1(W)$:

$$\sum_i [g^i \mid g] \otimes X_1, \qquad \partial_W X_1 = (g^{-1} - 1)(r).$$

This gives the following element of $C_1(G) \otimes_{\mathbb{Z}G} Z_1$:

$$[g] \otimes \sum_i g^{-i} \cdot X_1 \tag{3.34}$$

The element represented by (3.34) is a 1-cycle of (G, Z_1) whose class in $H_1(G, Z_1)$ is then the image of the class represented by (3.26). $c(g)$ in (3.31) is then the result of taking X_1 to be $-(g^{-1}(s), s)$ with $s = g(r)$. (3.32) follows by taking $X_1 = \sum_{0 \leq j < m} -h^{-j}(h^{-1}(s), s)$ with $s = h(r)$, and noting that X_1 is now h-invariant and $[h^m] = [g]$ is homologous to $m[h]$ for trivial coefficients. $\square$

3.35. Theorem. *$H_3(W/G^\delta) \cong \mathbb{Q}\{2\} \amalg$ 2-divisible group.*

Proof. As g ranges over the elements of finite order $n > 2$ in $\mathrm{SO}(2, \mathbb{R})$, the classes in $H_1(G, Z_1)$ arising as the images of the elements of order n form a subgroup of $H_1(G, Z_1)$. The compatibility argument described in the proof of Proposition 3.30 and the argument below imply that we have an injection of $\mathbb{Q}/\mathbb{Z}$ into $H_1(G, Z_1)$ through $H_3(\mathrm{SL}(2, \mathbb{R}), \mathbb{Z})$.

We consider the special case of $n = 4$. The element g can be taken to be the conjugate w_0 of the Weyl group element w. $c(w_0)$ is:

$$-[w_0] \otimes \sum_{0 \leq i < 4} w_0^i(r, s), \qquad s = w_0(r), \quad -\infty \leq r < s.$$

We will now determine the preimage of the class of $c(w_0)$ in $H_1(G, B_1)$ or $H_3(W/G^\delta)$. We are now forced to trace through the exact sequence (2.23) associated to (2.19). We already know that $w_0 = [x, y]$ as in (3.10). It follows that $c(w_0)$ is homologous to:

$$(3.36) \qquad [x]\otimes(y^{-1}-1)\cdot\sum_i w_0^i(r,s) - [y]\otimes(x^{-1}-1)\cdot\sum_i w_0^i(r,s).$$

To obtain this conclusion, we simply compute the boundary of

$$([w_0 \mid yx] - [x \mid y] + [y \mid x])\otimes \sum_i w_0^i(r,s).$$

We need to recall that the tensor product is over $\mathbb{Z}G$ and the second factor is w_0-invariant together with the fact that $w_0 yx = xy$.

A comparison of (3.36) and (3.13) shows that the image of $c(-1,-1)$ in $H_1(G, Z_1)$ is precisely 4 times a particular preimage of $c(w)$. Using the homomorphism L defined on $H_3(W/G^\delta)$, this shows that the preimages of the classes $c(g)$ as g ranges over elements of finite order greater than 2 in $\mathrm{SO}(2,\mathbb{R})$ defines a subgroup of $H_3(W/G^\delta)$ isomorphic to $\mathbb{Q}$ and contains the image of $c(-1,-1)$. The quotient group is isomorphic to $\mathbb{Q}/\mathbb{Z}$ and is identified with the torsion of $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ arising from the finite subgroups of $\mathrm{SL}(2,\mathbb{R})$. Since the image of $c(-1,-1)$ is $\pm 48\{2\}$, we get the desired assertion using the injectivity (=divisibility) of $\mathbb{Q}$ and Proposition 2.14. □

Remarks. In Dupont–Sah [4], the existence of the $\mathbb{Q}/\mathbb{Z}$ direct summand in $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ arising from the finite cyclic groups was derived using a variation of invariants found by Cheeger–Simons. The present argument bypasses some of the differential geometric ideas used. However, the use of Rogers' L-function is really nothing more than the Cheeger–Simons invariant in disguise.

In this connection, it should be noted that the argument used in pinpointing the image of $c(-1,-1)$ can be repeated to obtain preimages of $c(g)$ for $n>2$. It would then allow us to write $48\{2\}$ as integral linear combinations of other cross-ratio symbols $\{r\}$. A little bit of thought shows that these r's can be taken to be algebraic numbers in ℝ. Moreover, the coefficients can be arranged to have arbitrarily large integers as common divisors. This suggests some sort of analogue of the distribution relation found in Dupont–Sah [4]. We hope to return to this question elsewhere.

In view of the determination of L on the image of $c(-1,-1)$, we have a homomorphism:

$$(3.37) \qquad L_{\mathrm{red}} : H_1(G, Z_1) \to \mathbb{R}/4\pi^2\cdot\mathbb{Z}.$$

Its restriction to the image of $H_3(G,\mathbb{Z})$ then determines an element $L_{\mathrm{red}} \in H^3(\mathrm{SL}(2,\mathbb{R}), \mathbb{R}/4\pi^2\cdot\mathbb{Z})$. As pointed out in Gelfand–MacPherson [5], such an element has already been found by David Wigner [18]. There is a difference of a factor of 4 that comes about because Wigner's approach as outlined in Gelfand–MacPherson [5], see also Bloch [2], used the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$ while our approach uses the double cover of $\mathbb{P}^1(\mathbb{R})$. Moreover, Wigner's element is defined through the use of continuous cohomology theory and measurable cochains. By comparison, our approach is more elementary and it does not seem clear how to relate L_{red} directly to the first Pontrjagin class at this point.

Appendix A

We carry out the necessary modification of the calculations made in Dupont–Sah [4] in order to account for the difference between $SL(2,\mathbb{R})$ acting on the rays in $\mathbb{R}^2$ and $PSL(2,\mathbb{C})$ acting on the points of $\mathbb{P}^1(\mathbb{C})$. We note first that $-\infty \le r < \infty$ only parametrizes half of the rays. The other half requires the application of $-\mathrm{Id}$. The cross-ratio symbol $\{r\}$ stands for $(-\infty, 0, 1, r)$ in $C_3(W/G^\delta)$, $-\infty \le r < \infty$ and the latter satisfies the alternating rules (2.5) through (2.7).

In Dupont–Sah [4], $C_*^{\mathrm{bar}}(G, M)$ was the 'standard' normalized nonhomogeneous complex for G with coefficients in a left G-module M written in the form where $C_q^{\mathrm{bar}}(G, M)$ is generated by symbols:

$$[g_1 | \cdots | g_q]x, \qquad g_j \text{ in } G, \quad x \text{ in } M;$$

and

$$\partial_G [g_1 | \cdots | g_q]x = [g_2 | \cdots | g_q]x + \sum_{1 \le i \le q-1} (-1)^i [g_1 | \cdots | g_i g_{i+1} | \cdots | g_q] + (-1)^q [g_1 | \cdots | g_{q-1}] g_q(x).$$

These translate to our present $C_*(G) \otimes_{\mathbb{Z}G} M$ under the correspondence sending $[g_1 | \cdots | g_q]x$ to $[g_q^{-1} | \cdots | g_1^{-1}] \otimes x$ and ∂_G to $\varepsilon\partial_G \otimes \mathrm{Id}_M$.

As in Dupont–Sah [4], we have the following diagram of maps:

(A.1)
$$\begin{array}{ccccccc}
H_0(G, C_3) & \xrightarrow{\partial_W} & H_0(G, Z_2) \cong H_1(G, B_1) & \xrightarrow{i} & H_1(G, Z_1) & \longrightarrow & 0 \\
\| & & & & \big\downarrow d^2 & & \\
H_3(W/G^\delta) & & & & & & \\
& & \Lambda_{\mathbb{Z}}^2(A) \cong H_2(A, \mathbb{Z}) & \xleftarrow{p} & H_2(AN, \mathbb{Z}) \overset{\varrho}{\cong} H_2(G, C_0) & &
\end{array}$$

Here i and p are induced by inclusion on coefficients and projection on groups respectively. Splicing together (2.19) and (2.20), $l \circ \partial_W$ is really just d^2 in another spectral sequence. The kernel of i can be independently determined (see Theorem 3.24) and the kernel of d^2 in (A.1) is known to be the image of $H_3(G, \mathbb{Z})$ in $H_1(G, Z_1)$. It is therefore enough to know the composition of the maps from $H_0(G, C_3)$ to $\Lambda_{\mathbb{Z}}^2(A)$. This composition was determined in Dupont–Sah [4] for $PSL(2,\mathbb{C})$ acting on an acyclic chain complex based on distinct points of $\mathbb{P}^1(\mathbb{C})$. In that setting, the composition is simply d^3. For the present setting, we have $SL(2,\mathbb{R})$ acting on a chain complex with the integral homology of a circle.

We first recall the map ϱ from Dupont–Sah [4]. Let $\hat{}: G \to G$ denote any section to the projection map

$$G \to G/AN \cong S(\mathbb{R}^2) \cong K = SO(2, \mathbb{R}).$$

For $x_1, \ldots, x_q$, y in G, let $z_j = x_{j+1} \cdots x_q \cdot y$, $0 \le j \le q$. Then ϱ can be seen to be determined by the chain map (also denoted by ϱ):

(A.2)
$$\varrho([x_1 | \cdots | x_q] \cdot yAN) = [\hat{z}_0^{-1} x_1 \hat{z}_1 | \cdots | \hat{z}_{q-1}^{-1} x_q \hat{z}_q].$$

We now begin with $r>1$ in $\mathbb{R}$ and obtain

$$\begin{aligned}\partial_W\{r\} &= (0,1,r)-(-\infty,1,r)+(-\infty,0,r)-(-\infty,0,1)\\ &= (g_1-g_2+g_3-1)(-\infty,0,1)=\partial_G X_1 \qquad \text{(A.3)}\\ &\quad \text{where } X_1=[g_2g_1^{-1}]g_1(-\infty,0,1)-[g_3]\cdot(-\infty,0,1).\end{aligned}$$

Since $C_2(W)$ is $\mathbb{Z}G$-free with $(-\infty,0,1)$ as a generator, g_1, g_2 and g_3 are uniquely determined. However, X_1 is not unique. In order to simplify our notation, we note that $\mathrm{GL}^+(2,\mathbb{R})\cong \mathrm{SL}(2,\mathbb{R})\times\mathbb{R}^+\cdot\mathrm{Id}$ and $\mathbb{R}^+\cdot\mathrm{Id}$ acts trivially on the space $S(\mathbb{R}^2)$ of rays in $\mathbb{R}^2$. This allows us to represent elements of $G=\mathrm{SL}(2,\mathbb{R})$ by matrices in $\mathrm{GL}^+(2,\mathbb{R})$. In particular, we have

$$g_1=\begin{pmatrix}0 & r\\ 1-r & r\end{pmatrix},\quad g_2=\begin{pmatrix}r-1 & 1\\ 0 & 1\end{pmatrix},\quad g_3=\begin{pmatrix}r & 0\\ 0 & 1\end{pmatrix},\qquad r>1. \qquad \text{(A.4)}$$

We next have

$$\begin{aligned}&\partial_W X_1=\partial_G X_2,\\ &X_2=[g_3g_1g_2^{-1}\,|\,g_2g_1^{-1}](0,1)-[g_3g_2g_1^{-1}g_3^{-1}\,|\,g_3g_1g_2^{-1}](-\infty,1)\\ &\quad -[g_3g_2g_1^{-1}g_3^{-1}g_1g_2^{-1}\,|\,g_2g_1^{-1}](0,z)+[g_3g_2g_1^{-1}g_3^{-1}g_1g_2^{-1}\,|\,g_2^2](-\infty,0) \qquad \text{(A.5)}\\ &\quad -[g_2^2\,|\,g_3^2](-\infty,0)+[g_2g_1^{-1}\,|\,g_1^2](-\infty,0)-[g_1^2\,|\,g_3^{-1}](-\infty,0)\\ &\quad +[g_3^{-1}\,|\,g_3^2](-\infty,0).\end{aligned}$$

The determination of X_2 is formal and uses the following easily checked equalities (in $\mathrm{GL}^+(2,\mathbb{R})/\mathbb{R}^+\cdot\mathrm{Id}\cong\mathrm{SL}(2,\mathbb{R})$ as in Dupont-Sah [4]):

$$\begin{aligned}&g_1^{-1}g_3^{-1}g_1g_2=\begin{pmatrix}r(r-1) & 0\\ 0 & 1\end{pmatrix},\\ &g_2^{-2}g_3g_2=\begin{pmatrix}r & 0\\ 0 & r-1\end{pmatrix},\\ &g_3g_2g_1^{-1}g_3^{-1}g_1g_2^{-1}=g_2^2g_3^2g_2^{-2}, \qquad \text{(A.6)}\\ &g_1^{-1}g_2^{-1}g_1^2=g_3,\\ &g_2g_1^{-1}=g_1^2g_3^{-1}g_1^{-2}.\end{aligned}$$

In order to end up in $\Lambda^2_{\mathbb{Z}}(A)$, we need to apply ∂_W to X_2 in (A.5) and go through (A.2). We first note that $\partial_W X_2$ has 16 terms of the form:

$$[x\,|\,y]z(-\infty),\qquad x, y \text{ and } z\in G=\mathrm{SL}(2,\mathbb{R}).$$

According to (A.2), the ϱ image of such an element has the form:

$$[a\cdot b^{-1}\,|\,b\cdot c^{-1}],\qquad a=(x\hat{y}z)^{-1}\cdot xyz,\qquad b=(\hat{y}z)^{-1}\cdot yz,$$
$$c=\hat{z}^{-1}\cdot z,\qquad a,b,c\in AN.$$

It is straightforward to check that each of the 16 terms has the property that xyz, yz and z all map $-\infty$ into the set consisting of $-\infty, 0, 1, r$. We note that $(0) = w(-\infty)$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the Weyl group element. We can therefore select the section $^\wedge : G \to G$ to the projection $G \to G/AN$ so as to satisfy:

$$\hat{g} = \begin{cases} 1, & \text{if } g(-\infty) = (-\infty), \\ w, & \text{if } g(-\infty) = (0), \\ g_2 w, & \text{if } g(-\infty) = (1), \\ g_2^2 w, & \text{if } g(-\infty) = (r),\ 1 < r < \infty. \end{cases} \tag{A.7}$$

We note that (A.7) is only a partial section. It can be arbitrarily completed to a section. In view of the remarks already made, the computation as indicated depends only on (A.7) and does not depend how this partial section is completed. Now (A.7) is precisely the choice made in Dupont–Sah [4]. The rest of the arguments proceed just as in Dupont–Sah where the rather messy calculation as indicated in (A.2) is again omitted. The result is that the image of $\{r\}$ in $\Lambda^2_{\mathbb{Z}}(A)$ is simply $2 \cdot r \wedge (r-1)$, $r > 1$ in $\mathbb{R}$. We note that the expression in Dupont–Sah was written in the form of $2 \cdot z \wedge (1-z)$ in order to point out the connection with the defining relation for K_2 of a field. The factor of -1 is absorbed because both $\mathbb{R}^+$ and $\mathbb{C}^\times$ are divisible groups.

Remark. The section selected for (A.7) could just as well be selected by using Iwasawa decomposition. However, the resulting element appears more difficult to identify in the group $\Lambda^2_{\mathbb{Z}}(A)$.

Appendix B

We illustrate a result similar to the one sketched by Jeff Cheeger in his 1974 talk at the Vancouver congress.

B1. Proposition. *Both $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ and $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$ contain free abelian subgroups of countable infinite rank.*

Proof. We begin with the Artin–Schreier polynomials:

$$X^p - X - 1, \quad p \text{ a prime } > 3.$$

Reading this polynomial over $\mathbb{F}_p$, we conclude that it is irreducible over $\mathbb{Q}$ so that it has p distinct roots in $\mathbb{C}$. Since its derivative has two distinct real roots, each such polynomial can have at most three distinct real roots. Evidently, the largest real root r_p is strictly greater than 1 and has at least $p-3>0$ nonreal conjugates. We consider the cross-ratio symbol $\{r_p^{p-1}\}$. We have

$$\lambda\{r_p^{p-1}\} = r_p^{p-1} \wedge (r_p^{p-1} - 1) = r_p^{p-1} \wedge r_p^{-1} = 0.$$

Since λ is just $d^2 \circ \eta$ up to a factor of 2 in Theorem 2.35, $\eta\{r_p^{p-1}\}$ represents an element γ_p in $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$. We assert:

(B2) **The natural map of $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ into $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$ maps γ_p onto the free generators of a free abelian subgroup of countably infinite rank, p a prime >3.**

To see (B2), we recall the results in Dupont–Sah [4]. There the abelian group $\mathscr{P}_{\mathbb{C}}$ was defined by the generators $\{z\}$, $z \in \mathbb{C}-\{0,1\}$, with the defining relations corresponding to (2.4) without the restrictions on r_1, r_2. Similarly, the map λ is defined on $\mathscr{P}_{\mathbb{C}}$ and the kernel of λ is the quotient of $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$ modulo the trivial torsion subgroup $\mathbb{Q}/\mathbb{Z}$ arising from the finite cyclic subgroups of $\mathrm{SL}(2,\mathbb{C})$. We note that $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ operates on $\mathscr{P}_{\mathbb{C}}$ and that the complex version D of the dilogarithm defines a group homomorphism on $\mathscr{P}_{\mathbb{C}}$ so that $D\{z\}$ is simply the volume of the totally asymptotic 3-simplex in the extended hyperbolic 3-space with vertices on the boundary $\partial\mathscr{H}^3 = \mathbb{P}^1(\mathbb{C})$ given by $\infty, 0, 1, z$. This volume is 0 when z is real and nonzero when z is not real (and is positive or negative in accordance with the orientation).

Let k_n denote the algebraic number field generated by r_p over $\mathbb{Q}$ with p ranging over the first n primes greater than 3. Since each r_p has degree p over $\mathbb{Q}$, the degree of k_n over $\mathbb{Q}$ must be a multiple of each such p. It follows that the degree of k_n over $\mathbb{Q}$ is precisely $\prod_{1\le i\le n} p(i)$ and $r_{p(n)}$ has degree $p(n)$ over k_{n-1}. In particular, $r_{p(n)}$ has nonreal conjugates over the real field k_{n-1}. We can therefore find σ_n in $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ such that σ_n is the identity on k_{n-1} but $\sigma_n(r_{p(n)}^{p(n)-1})$ is not real because $r_{p(n)}$ and $r_{p(n)}^{p(n)-1}$ generate the same field of degree $p(n)$ over $\mathbb{Q}$. It follows that *the homomorphism* $D\circ\sigma_n$ is 0 on $\{r_p^{p-1}\}$ for $p=p(i)$, $1\le i<n$, and not 0 on $\{r_{p(n)}^{p(n)-1}\}$. This gives the independence of γ_p with p ranging over primes greater than 3. Since r_p is real, the symbol $\{r_p^{p-1}\}$ can be interpreted either in $H_3(W/G^\delta)$ or in $\mathscr{P}_{\mathbb{C}}$ and γ_p can be interpreted either in $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ or $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$. □

Remark. Outside of the trivial torsion $\mathbb{Q}/\mathbb{Z}$, we do not know the exact kernel of the homomorphism from $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$ to $H_3(\mathrm{SL}(2,\mathbb{C}),\mathbb{Z})$. It can be seen that the natural homomorphism from $H_3(W/G^\delta)$ to $\mathscr{P}_{\mathbb{C}}$ has kernel containing $\mathbb{Q}\{2\}$ and the image is known to be quite far from the fixed points of the complex conjugation map on $\mathscr{P}_{\mathbb{C}}$. Proposition B1 shows that the image is nevertheless quite large. Compare Dupont–Sah [4].

Actually, r_p is the unique real root of the Artin–Schreier polynomial exhibited as long as p is an odd prime. Other algebriac numbers can also be found with similar properties. r_2 is the golden ratio and leads to 5-torsion in $H_3(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$. We do not know much about $L\{r_p^{p-1}\}$, p odd prime. In view of (2.46), it seems reasonable to conjecture that they are $\mathbb{Q}$-linearly independent.

Appendix C

As indicated in Dupont–Sah [4], the distinct point complex based on the projective line $\mathbb{P}^1(F)$ over an arbitrary field F was first used by Bloch–Wigner (in unpublished private notes) to study the homology of $\mathrm{PGL}(2,F)$. Questions involving 2-torsion were bypassed through the use of coefficient rings containing 2 as a unit. This approach is evidently 'functorial' in F. When $F=\mathbb{R}$, we have the following exact sequence:

$$1 \to \pm I \to \mathrm{SL}^{\pm}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R}) \to 1. \tag{C.1}$$

Here $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is determined by the split exact sequence:

$$1 \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}^{\pm}(2,\mathbb{R}) \xrightarrow{\det} \pm 1 \to 1. \tag{C.2}$$

Our alternating chain complex C_* may be truncated through the replacement of C_4 by ∂C_4. The action of $\mathrm{SL}(2,\mathbb{R})$ on this truncated complex can then be extended to the group $\mathrm{SL}^{\pm}(2,\mathbb{R})$ in the obvious manner provided that both C_0 and $\mathbb{Z}$ are twisted by the determinant action through (C.2). However, this twisting of the coefficient group can cause drastic changes in the corresponding homology groups. In order to compare our results with the results in Dupont–Sah [4], we need to consider the distinct point complex based on $\mathbb{P}^1(\mathbb{R})$. Following the analysis of Dupont–Sah [4], it is quite easy to obtain the following exact sequence:

$$0 \to \mathbb{F}_2 \to H_3(\mathrm{PGL}(2,\mathbb{R})) \to \mathscr{P}_{\mathbb{R}} \to \Lambda^2_{\mathbb{Z}}(\mathbb{R}^+) \to H_2(\mathrm{PGL}(2,\mathbb{R})) \to \mathbb{F}_2 \to 0. \tag{C.3}$$

The first $\mathbb{F}_2$ in (C.3) is $H_3(\pm 1)$ and the splitting map is provided by the projection of $\mathrm{PGL}(2,\mathbb{R})$ onto ± 1 through (C.1) and (C.2). The second splitting map comes from the divisibility of $\Lambda^2_{\mathbb{Z}}(\mathbb{R}^+)$.

The abelian group $\mathscr{P}_{\mathbb{R}}$ is defined by generators $\{\{r\}\}$, $r \in \mathbb{R}-\{0,1\}$ with defining relations ($r_1 \neq r_2$ lie in $\mathbb{R}-\{0,1\}$):

$$\begin{aligned} R(r_1,r_2) = {} & \{\{r_1\}\} - \{\{r_2\}\} + \{\{r_2/r_1\}\} - \{\{(1-r_2)/(1-r_1)\}\} \\ & + \{\{(1-r_2^{-1})/(1-r_1^{-1})\}\}. \end{aligned} \tag{C.4}$$

As shown in Dupont–Sah [4, Lemma 5.7], $\mathscr{P}_{\mathbb{R}}$ is 2-divisible.

An examination of the Hochschild–Serre spectral sequence associated to the split exact sequence:

$$1 \to \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R}) \xrightarrow{\det} \pm 1 \to 1 \tag{C.5}$$

yields the following isomorphism and exact sequence:

$$H_2(\mathrm{PGL}(2,\mathbb{R})) \cong H_0(\pm 1, H_2(\mathrm{PSL}(2,\mathbb{R}))), \tag{C.6}$$

$$\mathbb{F}_2 \to H_0(\pm 1, H_3(\mathrm{PSL}(2,\mathbb{R}))) \to H_3(\mathrm{PGL}(2,\mathbb{R})) \to \mathbb{F}_2 \to 0. \tag{C.7}$$

As in Dupont–Sah [4], we can also analyze the Hochschild–Serre spectral sequence associated to the exact sequence:

(C.8) $\quad 1 \to \pm 1 \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}) \to 1.$

This then yields the exact sequences:

(C.9) $\quad 0 \to H_2(\mathrm{SL}(2,\mathbb{R})) \to H_2(\mathrm{PSL}(2,\mathbb{R})) \to \mathbb{F}_2 \to 0,$

(C.10) $\quad 0 \to \mathbb{Z}/4\mathbb{Z} \to H_3(\mathrm{SL}(2,\mathbb{R})) \to H_3(\mathrm{PSL}(2,\mathbb{R})) \to 0.$

The first map in (C.9) can be identified with multiplication by 2 on the group $K_2^0(\mathbb{R}) \amalg \mathbb{Z} \cdot c(-1,-1)$. Since $K_2^0(\mathbb{R})$ is the image of $\Lambda_{\mathbb{Z}}^2(\mathbb{R}^+)$ with $\mathbb{R}^+$ identified with A and ± 1 in (C.5) can be covered by diagonal matrices, it is immediate that ± 1 acts trivially on $K_2^0(\mathbb{R})$. Since $\mathbb{Z} \cdot c(-1,-1)$ corresponds to the fundamental group of $K \cong \mathrm{SO}(2)$ and ± 1 can also be covered by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ inverting SO(2), it is clear that ± 1 inverts $\mathbb{Z} \cdot c(-1,-1)$. It follows that:

(C.11) $\quad H_2(\mathrm{PGL}(2,\mathbb{R})) \cong K_2^0(\mathbb{R}) \amalg \mathbb{F}_2 \cong K_2(\mathbb{R}).$

The subgroup $\mathbb{Z}/4\mathbb{Z}$ in (C.10) is mapped onto the subgroup of order 4 in $H_3(\mathrm{SL}(2,\mathbb{R}))$ arising from the inclusion of the cyclic subgroup of order 4 generated by $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, see Dupont–Sah [4]. It follows that (2.36) can be rephrased as the exact sequence:

(C.12) $\quad 0 \to H_3(\mathrm{PSL}(2,\mathbb{R})) \to \mathscr{P}_{\mathbb{R}}^{\mathrm{PS}} \to \Lambda_{\mathbb{Z}}^2(\mathbb{R}^+) \to K_2^0(\mathbb{R}) \to 0$

Here $\mathscr{P}_{\mathbb{R}}^{\mathrm{PS}}$ is the abelian group generated by the symbols $\{\{r\}\}$, $r \in \mathbb{R}$, $r > 1$, with defining relations $R(r_1, r_2)$ in (C.4) subject to the restrictions that $1 < r_1 < r_2$ as in (2.4) and the additional relation of $12 \cdot \{\{2\}\}$.

(C.3) can be rephrased with the help of (C.7) and (C.10) in the form of the following exact sequence:

(C.13) $\quad 0 \to \mathrm{im}(H_0(\pm 1, H_3(\mathrm{PSL}(2,\mathbb{R})))) \to \mathscr{P}_{\mathbb{R}} \to \Lambda_{\mathbb{Z}}^2(\mathbb{R}^+) \to K_2^0(\mathbb{R}) \to 0.$

As shown in Dupont–Sah [4, Lemma 5.5], $12 \cdot \{\{2\}\}$ represents 0 in $\mathscr{P}_{\mathbb{R}}$. We therefore have a natural homomorphism from $\mathscr{P}_{\mathbb{R}}^{\mathrm{PS}}$ to $\mathscr{P}_{\mathbb{R}}$ sending each generator $\{\{r\}\}$, $r > 1$, of $\mathscr{P}_{\mathbb{R}}^{\mathrm{PS}}$ to the corresponding generator $\{\{r\}\}$ of $\mathscr{P}_{\mathbb{R}}$.

C.14. Theorem. *The natural homomorphism from* $\mathscr{P}_{\mathbb{R}}^{\mathrm{PS}}$ *to* $\mathscr{P}_{\mathbb{R}}$ *is an isomorphism. In particular,* $H_3(\mathrm{PSL}(2,\mathbb{R})) \cong H_0(\pm 1, H_3(\mathrm{PSL}(2,\mathbb{R})))$ *under the natural map. In other words,* ± 1 *acts trivially on* $H_3(\mathrm{PSL}(2,\mathbb{R}))$ *and* (C.7) *can be replaced by:*

$$H_3(\mathrm{PGL}(2,\mathbb{R})) \cong H_3(\mathrm{PSL}(2,\mathbb{R})) \amalg \mathbb{F}_2.$$

Proof. A number of the arguments are already present in Dupont–Sah [4, Lemmas 5.4, 5.5, and 5.7]. We reorganize them. Let $C = \amalg_{r \neq 0,1} \mathbb{Z} \cdot \{\{r\}\}$ with subgroup $C' = \amalg_{r>1} \mathbb{Z} \cdot \{\{r\}\}$, $r \in \mathbb{R}$. Let B be the subgroup of C generated by $R(r_1, r_2)$ in (C.4) with $r_1 \neq r_2$ ranging over $\mathbb{R} - \{0, 1\}$ and let $B' \subset C' \cap B$ be the subgroup generated by $R(r_1, r_2)$ in (C.4) with $r_2 > r_1 > 1$. By definition, $\mathscr{P}_{\mathbb{R}} = C/B$, and $H_3(W/G^\delta) \cong C'/B'$ (see Theorem 2.35).

In accordance with Dupont–Sah [4, Lemma 5.4, 5.5], we have a subgroup D of B generated by $4\{\{-1\}\}$ and $R_i(r)$, $1 \le i \le 4$, defined by

$$\text{(C.15)}\quad \begin{aligned} &R_1(r)=\{\{r\}\}+\{\{r^{-1}\}\}, \quad r>0 \text{ and } r\neq 1;\\ &R_2(r)=\{\{r\}\}+\{\{r^{-1}\}\}-2\{\{-1\}\}, \quad r<0;\\ &R_3(r)=\{\{r\}\}+\{\{1-r\}\}-2\{\{1/2\}\}, \quad 0<r<1;\\ &R_4(r)=\{\{r\}\}+\{\{1-r\}\}-2\{\{1/2\}\}+2\{\{-1\}\}, \quad \text{either } r<0 \text{ or } r>1.\end{aligned}$$

It is then easy to conclude that the natural map in Theorem C.14 is surjective. Straightforward computations imply that ($r_1 \neq r_2$ in $\mathbb{R}-\{0,1\}$):

$$\text{(C.16)}\qquad R(r_2,r_1),\ R(r_1^{-1},r_2^{-1}),\ R(1-r_1,1-r_2),\ R(r_2/r_1,r_2)\equiv \pm R(r_1,r_2) \mod D.$$

We next assert that:

$$\text{(C.17)}\qquad B=B'+D.$$

It is enough to show that $R(r_1,r_2)\in B'+D$ for $r_1\neq r_2$ in $\mathbb{R}-\{0,1\}$. Since

$$R(r_1,r_2)\equiv R(1-r_1,1-r_2) \mod D,$$

we can assume $r_1>0$. Similarly, using $R(r_1^{-1},r_2^{-1})$, we may assume $0<r_1<1$; and using $R(1-r_1,1-r_2)$ once more, we may also assume $r_2>0$. If $0<r_2<1$, then either $R(r_1^{-1},r_2^{-1})$ or $R(r_2^{-1},r_1^{-1})\in B'$. If $r_2>1$, then either $R(r_2/r_1,r_2)$ or $R(r_2,r_2/r_1)\in B'$. (C.17) therefore follows from (C.16).

The injectivity assertion in Theorem C.14 is equivalent with: $B\cap C'=B'+\mathbb{Z}\cdot 12\{\{2\}\}$. Using (C.17), it is then equivalent with:

$$\text{(C.18)}\qquad D\cap C'=D\cap B'+\mathbb{Z}\cdot 12\{\{2\}\}.$$

In order to show (C.18), we note that for $r>2$, we have:

$$\text{(C.19)}\quad \begin{aligned} &R_1(r)-R_3(r^{-1})+R_1(r/(r-1))-2R_1(2)=R(r/(r-1),2)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad -R(2,r)\in B'\cap D,\\ &R_4(r)=R_1(r)-R_3(r^{-1})+R_1(r/(r-1))-2R_1(2)-R_4(r/(r-1))\\ &\qquad +R_2(1-r)+2R_4(2).\end{aligned}$$

From these follows the assertion:

(C.20) $\quad D$ can be defined with $R_4(r)$ restricted so that $-1\le r<0$ or $1<r\le 2$.

With (C.20) in force, suppose that an element x of C' is expressible as an integral linear combination of $4\{\{-1\}\}$ and $R_i(r)$, $1\le i\le 4$, i.e., $x\in D\cap C'$. Since no symbols $\{\{r\}\}$, $r<-1$, appear in x, (C.20) and the definition of D and C' shows that there are no $R_2(r)$-terms in our integral linear combination. Since no symbols $\{\{r\}\}$, $r<0$, appear in x, the preceding assertion implies that there are no $R_4(r)$-terms other than $R_4(2)=R_4(-1)$ in our integral linear combination. Since we are only interested in showing that x lies in $D\cap B'+\mathbb{Z}\cdot 12\{\{2\}\}$, the first identity in

(C.19) now allows us to assume that x is expressible as an integral linear combination of $4\{\{-1\}\}$, $R_4(2)$, and $R_1(r)$. Since no symbols $\{\{r\}\}$, $r<1$, appear in x, our integral linear combination can only involve $4\{\{-1\}\}$, $R_4(2)$ and $R_1(2)=R_1(1/2)$. It is then clear that x can be expressed as an integral linear combination of $3\{\{2\}\}+3\{\{-1\}\}$ and $4\{\{-1\}\}$. Since $\{\{-1\}\}$ does not appear in x, we conclude that x must lie in $\mathbb{Z}\cdot 12\{\{2\}\}$ as asserted in (C.18).

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